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Predicting low-dimensional spatiotemporal dynamics using discrete wavelet transforms

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A method is presented for predicting spatiotemporal time series whose dynamics is generated by a low-dimensional deterministic dynamical system. It is based on a combination of time delay embedding and wavelet expansion and is also applicable in cases where the dynamics may not be linearly decomposed into the evolution of a small number of spatial modes. As an example, we predict chaotic transversal motions of two Gaussian pulses along a one-dimensional axis.

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Methods for analyzing and predicting spatiotemporal time series (STTS) are currently one of the most important challenges in nonlinear time series analysis [1,2]. In general, a STTS consists of a sequence of (spatial) arrays $\mathbf{x}^n \in \mathbb{R}^M$ that are taken from a spatially extended dynamical system at discrete times $t_n = n\Delta t$ $(n=1,\dots,N)$, where Δt is the sampling time and N is the length of the time series, i.e., the total number of patterns. Each frame $\mathbf{x}^n$ consists of M spatial samples x_m^n or "pixels" and may thus (at a fixed time n) formally be considered as a vector in $\mathbb{R}^M$.

The most successful approach developed until now to analyze STTS $\{\mathbf{x}^n\}$ is based on a linear decomposition of the dynamics into spatial modes $\mathbf{b}^k$. The goal of this approach is to find a few dominating modes $\mathbf{b}^1,\dots,\mathbf{b}^K$ $(K \ll M)$ so that the patterns $\mathbf{y}^n = \mathbf{a} + \sum_{k=1}^K c_k^n \mathbf{b}^k$ provide a good approximation of the sequence $\{\mathbf{x}^n\}$. The vectors $\mathbf{a}$ and $\mathbf{b}^k$ are constant and may be computed using the Karhunen-Loève transformation, for example [3–9]. The information about the temporal evolution is contained in the coefficients c_k^n. If such a decomposition into a few dominating modes is possible this method turns out to be very efficient. From the temporal evolution of the (few) coefficients $\{c_k^n\}$ $(k=1,\dots,K)$ low-dimensional models can be derived and may be used for analyzing and predicting the STTS [3–10].

However, even for low-dimensional systems the number of dominating modes of a Karhunen-Loève transformation may be arbitrarily large. The dynamics of a δ pulse, for example, moving periodically back and forth on a one-dimensional axis, can only be covered by as many basis vectors as different positions of the pulse occur. Its closed orbit in state space is one dimensional but a large number of modes $\mathbf{b}^k$ is necessary to describe the STTS. This problem occurs in general when moving structures (e.g., solitons) appear in a spatially extended system. Our numerical example given below therefore consists of a localized pulse that moves chaotically on a one-dimensional axis.

Experimental examples for a spatially extended system with moving localized structures are cavitation bubble fields in an external sound field [11,12]. The dynamics of the bubbles can be described as oscillation in size and convection in space. Additionally, it can be observed that filamentary structures are generated as substructures. Since wavelets are objects that are localized in both real space and Fourier space, they appear to be the natural choice for basis functions for localized structures of a well defined size or scale. The basic physical mechanism underlying this pattern formation process may be modeled using a system of partial differential equations (and is a topic of current research) [13,14]. Experimental investigations of the global sound field emitted by the oscillating bubbles give evidence for a low-dimensional strange attractor [11,15–17]. Therefore the prediction method presented in this paper presents itself as a possible tool for predicting and modeling the spatiotemporal dynamics of cavitation bubble fields. This is of interest not only for basic research but also for many applications where ultrasound of high intensity is used.

For STTS whose dynamics originates from a low-dimensional chaotic attractor but cannot be decomposed into a small number of spatial modes we suggest the following prediction method. Firstly, we make use of the fact that the dynamics is low dimensional and reconstruct the state space using time delay coordinates $\mathbf{u}^n = (h^n, h^{n+l}, \dots, h^{n+(d-1)l})$ from a suitable *global* observable $h^n = h(\mathbf{x}^n)$ of the system, e.g., some mean value of a physical quantity. The dimension d and the time delay l (lag) of this reconstruction have, of course, to be chosen suitably [1,2]. Each state vector $\mathbf{u}^n$ corresponds then uniquely to a pattern $\mathbf{x}^n$. If we want to predict the temporal evolution of a given state $\mathbf{u}$ we determine the J nearest neighbors $\mathbf{u}^{n_j}$ $(j=1,\dots,J)$ of $\mathbf{u}$ in state space and superimpose the corresponding future patterns $\mathbf{x}^{n_j+1}$. The details of this superposition determine the accuracy of the prediction. For our numerical example we will use a weighted sum

$$\mathbf{x}^{n+1} = \sum_{j=1}^{J} w_j^n \mathbf{x}^{n_j+1} \tag{1}$$

with weights given by

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$$w_j^n = \frac{e_j^n}{\sum_{j=1}^{J} e_j^n}, \quad (2)$$

where

$$e_j^n = \exp\left(- \frac{\|\mathbf{u} - \mathbf{u}^{n_j}\|^2}{s^2} \right). \quad (3)$$

For the parameter s we choose $s = \max_{j=1,..,J} \|\mathbf{u} - \mathbf{u}^{n_j}\|$. To use the prediction method in this form the neighboring patterns $\mathbf{x}^{n_j}$ have to be easily accessible. If they are large ($M \gg 10\,000$, for example) this may lead to considerable storage and CPU-time requirements if long sequences ($N \gg 1000$) are used. To avoid this problem one may use the above-introduced scheme in combination with a Galerkin expansion method based on discrete wavelet transforms (DWT's) [18]. The DWT of many patterns consists only of a few dominating coefficients whose value and location in the spectrum have to be stored. This truncation typically yields compression rates between 10 and 100 and may also serve for noise reduction. Furthermore, since the DWT is a linear transform, the superposition (1) may be computed in the wavelet domain for the dominating wavelet coefficients only and then be subjected to the inverse DWT to obtain the predicted pattern. If the wavelet transform is performed at the beginning of the analysis the global observable h may be defined as a function of the wavelet coefficients. In this case the position and the scale of spatial structures can be taken into account very efficiently.

To demonstrate the efficiency of the proposed prediction scheme we have generated a STTS $\{\mathbf{x}^n\}$ that consists of two Gaussian pulses that move chaotically on a one-dimensional

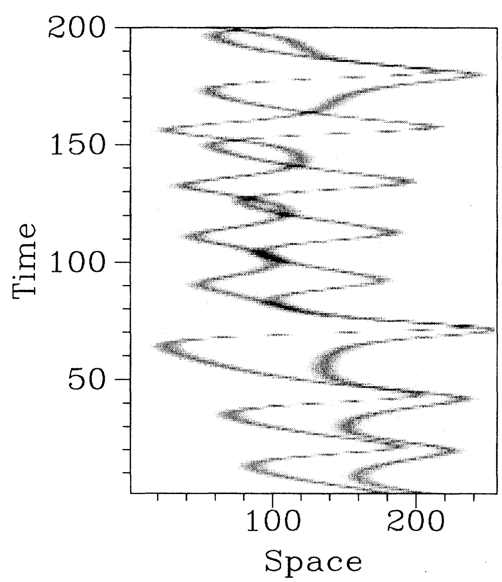

FIG. 1. Spatiotemporal evolution of a Gaussian pulse whose position is controlled by the Lorenz attractor to simulate a chaotically moving structure. Shown are 200 time steps with a sampling rate of $\Delta t = 0.02$. The spatial resolution is $M = 256$ and the noise amplitude $\varepsilon = 0.1$.

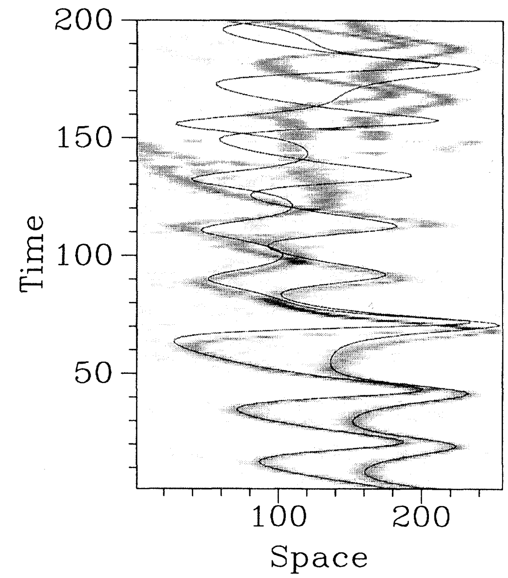

FIG. 2. Result of an iterated nonlinear prediction of the STTS shown in Fig. 1. For the reconstruction the ten dominating Lemarie wavelet coefficients were used. The solid lines give the true positions of the centers of the pulses. To appreciate the effect of the wavelet truncation only, compare Fig. 3.

axis. The position of the Gaussian pulses is controlled by the x variable and the z variable of the Lorenz system

$$\dot{x} = -\sigma x + \sigma y, \quad (4)$$

$$\dot{y} = -xz + Rx - y, \quad (5)$$

$$\dot{z} = xy - bz, \quad (6)$$

with $R = 45.92$, $\sigma = 16$, and $b = 4$. The sampling rate was $\Delta t = 0.02$ and the spatial resolution $M = 256$. Uniformly dis-

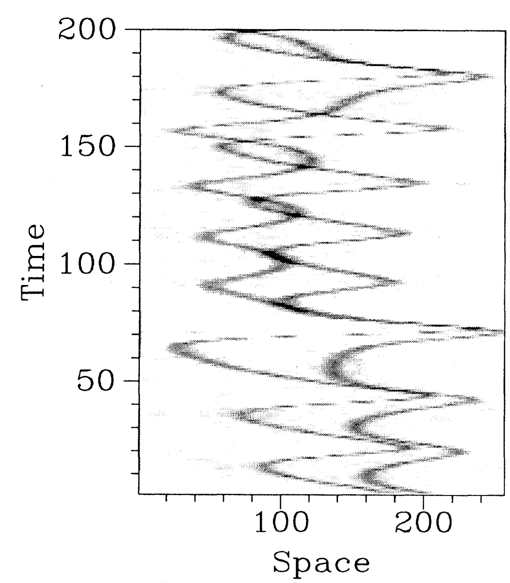

FIG. 3. Time series shown in Fig. 1 truncated using the ten dominating Lemarie wavelet coefficients only (without prediction).

tributed random numbers $r \in [-\varepsilon,\varepsilon]$ with $\varepsilon=0.1$ were added to the STTS to simulate measurement noise. Two hundred time steps of the resulting noisy STTS are shown in Fig. 1. As a global observable h we used the center of mass of the (spatial) patterns, i.e., $h^n = h(\mathbf{x}^n) = \Sigma_{m=1}^{M} m x_m^n / \Sigma_{m=1}^{M} x_m^n$. The states were reconstructed in a four-dimensional state space using a time delay of $l=5$. For the local fits approximating the flow in the reconstructed state space we used $J=4$ nearest neighbors [see Eq. (1)]. The neighboring state points were taken from a sequence of 4000 patterns of the STTS that was used as a training set for the prediction scheme. Figure 2 gives the result of a nonlinear prediction that was computed iteratively, i.e., for the 200 time steps shown we always used the last predicted state as the new reference point for the next prediction step. The true positions of the centers of the two pulses are indicated by solid curves in Fig. 2. For projecting the spatial patterns onto dominant subspaces we used DWT's based on Lemarie wavelets [18]. Only the ten largest wavelet coefficients of each pattern $\mathbf{x}^n$ were stored and used for reconstruction corresponding to a compression ratio of more than 25. To visualize the effect of this truncation Fig. 3 shows the STTS given in Fig. 1 that was subjected to the DWT and then reconstructed using the ten largest coefficients only. To appreciate the efficiency of the prediction scheme one should compare the predicted (and truncated) STTS shown in Fig. 2 to the truncated original time series given in Fig. 3. The good agreement of both patterns during the first half of the time interval shows that the method allows efficient intermediate time predictions of STTS.

In conclusion, we have presented a prediction scheme for low-dimensional spatiotemporal time series that is based on a time-dependent activation of spatial patterns (or modes) using a low-dimensional state space reconstruction from a suitably chosen global observable of the system. The basic idea of this method was demonstrated using a locally constant prediction scheme in state space and a discrete wavelet representation of the spatial patterns. Generalizations using more sophisticated prediction schemes (e.g., radial basis functions or neural networks) and different encodings of the spatial patterns to be activated will be discussed elsewhere.

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